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LETTER TO THE EDITOR

Representations of $\mathcal{U}_q(\hat{A}_N)$ in the space of continuous anyons

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Abstract. Representations of the quantum algebra $\mathcal{U}_q(\hat{A}_N)$ are constructed on the space of $(N+1)$-component anyons in $\mathbb{R}$, extending analogous results on the lattice. Such representations can be obtained in terms of both fermionic or bosonic anyons, showing that the hard-core constraint is not necessary in the continuous case.

In the last few years much attention has been drawn to an enlarged concept of symmetry represented by quantum algebras. It has been found that a great number of areas, from conformal physics to integrable models, display such kind of underlying symmetries. The common belief that the so-called generalized statistics are strictly tied with quantum algebras has been confirmed in [1, 2]. In [2] a representation of the quantum algebra $SU_q(2)$ has been explicity realized in terms of anyon creation and annihilation operators on the lattice. The construction turns out to be a very natural generalization of the well known Schwinger representation of $A_1 = SU(2)$. As fermions are continuously deformed into anyons, by varying the statistical parameter from zero to a generic $\vartheta \in \mathbb{R}$, the $SU(2)$ Schwinger representation gets deformed into $\mathcal{U}_q(A_1) = SU_q(2)$, the deformation parameter q being related to the statistical parameter ϑ by

$$q = \exp(\mathrm{i}\pi\vartheta)\,. \tag{1}$$

Such a construction has been further extended [3, 4] to the other non-exceptional quantum algebras series $\mathcal{U}_q(A_N), \mathcal{U}_q(B_N), \mathcal{U}_q(C_N), \mathcal{U}_q(D_N)$, and recently [5] to their affine version $\mathcal{U}_q(\hat{A}_N), \mathcal{U}_q(\hat{B}_N), \mathcal{U}_q(\hat{C}_N), \mathcal{U}_q(\hat{D}_N)$. These constructions clearly show that anyons are deeply related with quantum algebras, just as fermions (and bosons) are related with classical Lie algebras. A technical, but very important, point in these representations is that they are defined with anyons living on a lattice, so if one wishes to go over to the continuum, one must take a limit which, as a rule, presents some difficulties. Moreover, only the so called 'fermionic' anyons have been adopted so far. They satisfy the Pauli exclusion principle, stating that no more then one anyon with given quantum numbers can be present

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at each lattice site. In this letter we construct representations of the quantum algebras $\mathcal{U}_q(A_N), \mathcal{U}_q(\hat{A}_N)$ in the spirit of [5], but directly in the Fock space of anyons in $\mathbb{R}$, both of fermionic or bosonic type. As already observed in [2], if the rising step operators are realized in terms of anyons with statistical parameter ϑ, the corresponding lowering step operators must be realized with anyons whose statistical parameter is $-\vartheta$. Consequently, our first goal will be to realize a common Fock space for anyons with both statistical parameters, which we may consider to be mutually parity conjugate. Once this problem has been solved, we will use the annihilation and creation operators to construct our representations. These representations contain some arbitrary continuous parameters, which play the role of displacement in the wavefunction, and thus mix the inner space and the space of coordinates.

Let us describe briefly the basic objects we will need in what follows. For convenience we introduce the parameter η which takes the value $+1$ and -1 for bosonic and fermionic anyons, respectively. The anyon exchange relations then read

$$a_\mu(x_1)\, a_\nu(x_2) - \eta\, q^{\varepsilon(x_2-x_1)\delta_{\mu\nu}}\, a_\nu(x_2)\, a_\mu(x_1) = 0 \tag{2}$$

$$a^{*\mu}(x_1)\, a^{*\nu}(x_2) - \eta\, q^{\varepsilon(x_2-x_1)\delta_{\mu\nu}} a^{*\nu}(x_2)\, a^{*\mu}(x_1) = 0 \tag{3}$$

$$a_\mu(x_1)\, a^{*\nu}(x_2) - \eta\, q^{-\varepsilon(x_2-x_1)\delta_{\mu\nu}} a^{*\nu}(x_2)\, a_\mu\ (x_1) = \delta^\nu_\mu\, \delta(x_1 - x_2)\,. \tag{4}$$

Here $x_1, x_2 \in \mathbb{R}$, $\varepsilon(x)$ is the sign function, which we assume to vanish at zero, and q is a constant phase given by (1). The indices μ, ν range from 1 to N. Here and in what follows no sum convention is assumed unless explicitly stated. Even integer values of ϑ reproduce bosons when $\eta = 1$ and fermions when $\eta = -1$. The Fock representation for anyon fields strongly resembles that for CCR and CAR. As a one-particle Hilbert space one takes

$$\mathcal{H} = \bigoplus_{a=1}^{N} L^2(\mathbb{R}, \mathrm{d}x)\,. \tag{5}$$

The elements of $f \in \mathcal{H}$ will be represented as columns with N components. The scalar product is

$$(f, g) = \int \mathrm{d}x\ \sum_{\mu=1}^{N} f^{\dagger\mu}(x) g_\mu(x) = \sum_{\mu=1}^{N} \int \mathrm{d}x\ \overline{f}_\mu(x) g_\mu(x) \tag{6}$$

where $\dagger$ stands for Hermitian conjugation. In this notation an n-particle wavefunction $\varphi \in \mathcal{H}^n$ is a column whose entries are $\varphi_{\mu_1\ldots\mu_n}(x_1, \ldots, x_n)$. The direct sum

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n \tag{7}$$

where $\mathcal{H}^0 = \mathbb{C}^1$, is called the Fock space over $\mathcal{H}$. The elements of $\mathcal{F}(\mathcal{H})$ can be represented by sequences $\{\varphi = (\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(n)}, \ldots) :\ \varphi^{(n)} \in \mathcal{H}^n\}$ and the finite particle subspace $\mathcal{F}^0(\mathcal{H}) \subset \mathcal{F}(\mathcal{H})$ is defined as follows: $\varphi \in \mathcal{F}^0(\mathcal{H})$ if and only if $\varphi^{(n)} = 0$ for n large enough. By construction $\mathcal{F}^0(\mathcal{H})$ is dense in $\mathcal{F}(\mathcal{H})$.

Let us define the subspace $\mathcal{H}^n_R$ of the functions in $\mathcal{H}^n$ having the following exchange property:

$$\begin{aligned}&\varphi_{\mu_1\ldots\mu_i\mu_{i+1}\ldots\mu_n}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n)\\ &\qquad = \eta\, q^{\varepsilon(x_i-x_{i+1})\delta_{\mu_i\mu_{i+1}}} \varphi_{\mu_1\ldots\mu_{i+1}\mu_i\ldots\mu_n}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)\,.\end{aligned} \tag{8}$$

Setting $\mathcal{H}^0_R = \mathcal{H}^0$ and $\mathcal{H}^1_R = \mathcal{H}^1$ we define

$$\mathcal{F}_R(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n_R\,. \tag{9}$$

We denote by $\mathcal{F}^0_R(\mathcal{H})$ the corresponding dense subspace with finite particles. On $\mathcal{F}^0_R(\mathcal{H})$ one defines the annihilation and creation operators as follows [6]:

$$\left[a(f)\varphi\right]^{(n)}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n) = \sqrt{n+1}\int \mathrm{d}x \sum_{\mu_0=1}^{N} f^{\dagger\mu_0}(x)\varphi^{(n+1)}_{\mu_0\mu_1\ldots\mu_n}(x,x_1,\ldots,x_n) \tag{10}$$

$$\begin{aligned}\left[a^*(f)\varphi\right]^{(n)}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n) &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} \eta^{k-1}\, q^{\sum_{j=1}^{k}\varepsilon(x_j-x_k)\delta_{\mu_j\mu_k}} f_{\mu_k}(x_k)\\ &\quad\times\varphi^{(n-1)}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_n}(x_1,\ldots,\widehat{x}_k,\ldots,x_n)\,.\end{aligned} \tag{11}$$

Introducing the operator-valued distributions $a_\alpha(x)$ and $a^{*\alpha}(x)$ defined by

$$a(f) = \int \mathrm{d}x \sum_{\alpha=1}^{N} f^{\dagger\alpha}(x)a_\alpha(x) \qquad a^*(f) = \int \mathrm{d}x \sum_{\alpha=1}^{N} f_\alpha(x)a^{*\alpha}(x) \tag{12}$$

one can indeed verify that the exchange relations (2)–(4) are satisfied. On the same space $\mathcal{F}^0_R(\mathcal{H})$ we also define $\tilde{a}^*(f), \tilde{a}(f)$ as

$$\begin{aligned}\left[\tilde{a}(f)\varphi\right]^{(n)}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n) &= \sqrt{n+1}\;\eta^n \int \mathrm{d}x \sum_{\mu_0=1}^{N} q^{-\sum_{j=1}^{n}\varepsilon(x-x_j)\delta_{\mu_0\mu_j}} f^{\dagger\mu_0}(x)\\ &\quad\times\varphi^{(n+1)}_{\mu_0\mu_1\ldots\mu_n}(x,x_1,\ldots,x_n)\end{aligned} \tag{13}$$

$$\begin{aligned}\left[\tilde{a}^*(f)\varphi\right]^{(n)}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n) &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} \eta^{n-k}\, q^{\sum_{j=k}^{n}\varepsilon(x_k-x_j)\delta_{\mu_k\mu_j}} f_{\mu_k}(x_k)\\ &\quad\times\varphi^{(n-1)}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_n}(x_1,\ldots,\widehat{x}_k,\ldots,x_n)\,.\end{aligned} \tag{14}$$

Direct computation shows that $\tilde{a}^{*\mu}(x), \tilde{a}_\mu(x)$ satisfy the exchange relations (2)–(4) with q replaced by q^{-1}, i.e. with ϑ replaced by $-\vartheta$.

The quantum affine algebra $\mathcal{U}_q(\hat{A}_{N-1})$ is an associative algebra over $\mathbb{C}$ with identity $\mathbb{1}$, whose generators $\{e_\alpha, f_\alpha, h_\alpha \,:\, \alpha = 0,1,\ldots,N-1\}$ satisfy the commutation rules

$$\begin{aligned} &[h_\alpha, e_\beta] = a_{\alpha\beta}e_\beta \qquad && [h_\alpha, h_\beta] = 0\\ &[h_\alpha, f_\beta] = -a_{\alpha\beta}f_\beta \qquad && [e_\alpha, f_\beta] = \delta_{\alpha\beta}[h_\alpha]_q \end{aligned} \tag{15}$$

along with the quantum Serre relations

$$\begin{aligned} &\sum_{k=0}^{1-a_{\alpha\beta}} (-1)^k \begin{bmatrix} 1-a_{\alpha\beta} \\ k \end{bmatrix}_q (e_\alpha)^{1-a_{\alpha\beta}-k} e_\beta (e_\alpha)^k = 0\\ &\sum_{k=0}^{1-a_{\alpha\beta}} (-1)^k \begin{bmatrix} 1-a_{\alpha\beta} \\ k \end{bmatrix}_q (f_\alpha)^{1-a_{\alpha\beta}-k} f_\beta (f_\alpha)^k = 0\,. \end{aligned} \tag{16}$$

In equations (16) $a_{\alpha\beta}$ is the extended Cartan matrix of $\hat{A}_{N-1}$, and we used the common notation

$$[a]_q = \frac{q^a - q^{-a}}{q-q^{-1}} \qquad \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!} \qquad [m]_q! = [1]_q[2]_q\cdots[m]_q\,. \tag{17}$$

If we remove the value $\alpha = 0$, equations (15), (16) become the defining relations of the quantum algebra $\mathcal{U}_q(A_{N-1})$, which is therefore automatically included in our treatment. The algebra $\mathcal{U}_q(\hat{A}_{N-1})$ has a central element (charge) γ, which is given by

$$\gamma = \sum_{\alpha=0}^{N-1} h_\alpha \,. \tag{18}$$

It is well known that $\mathcal{U}_q(\hat{A}_{N-1})$ is endowed with a co-product $\Delta : \mathcal{U}_q(\hat{A}_{N-1}) \to \mathcal{U}_q(\hat{A}_{N-1}) \otimes \mathcal{U}_q(\hat{A}_{N-1})$, given by

$$\begin{aligned} \Delta(h_\alpha) &= h_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes h_\alpha \\ \Delta(e_\alpha) &= e_\alpha \otimes \mathbb{1} + q^{h_\alpha} \otimes e_\alpha \\ \Delta(f_\alpha) &= f_\alpha \otimes q^{-h_\alpha} + \mathbb{1} \otimes w f_\alpha \,. \end{aligned} \tag{19}$$

It is also useful to define, by recurrence, the n-fold nested co-products $\Delta^{(n)} : \mathcal{U}_q(\hat{A}_{N-1}) \to \mathcal{U}_q(\hat{A}_{N-1})^{\otimes n}$:

$$\begin{aligned} &\Delta^{(1)} = id \qquad \Delta^{(2)} = \Delta \\ &\Delta^{(n+1)} = (\Delta \otimes id^{\otimes(n-1)}) \circ \Delta^{(n)} \,. \end{aligned} \tag{20}$$

Using the N-component anyons introduced above, we shall construct a representation of the quantum affine algebra $\mathcal{U}_q(\hat{A}_{N-1})$ with zero central charge; in particular, in this way we will also obtain a representation of the quantum algebra $\mathcal{U}_q(A_{N-1})$. Consider the fundamental representation of $\mathcal{U}_q(\hat{A}_{N-1})$, given by

$$e_\alpha \mapsto E_\alpha \qquad f_\alpha \mapsto F_\alpha \qquad h_\alpha \mapsto H_\alpha \tag{21}$$

where

$$\begin{aligned} &(E_i)_\mu^\nu = \delta_{\mu\, i}\delta^{\nu\, i+1} && (E_0)_\mu^\nu = \delta_{\mu\, N}\delta^{\nu\, 1} \\ &(F_i)_\mu^\nu = \delta_{\mu\, i+1}\delta^{\nu\, i} && (F_0)_\mu^\nu = \delta_{\mu\, 1}\delta^{\nu\, N} \\ &(H_i)_\mu^\nu = \delta_{\mu\, i}\delta^{\nu\, i} - \delta_{\mu\, i+1}\delta^{\nu\, i+1} && (H_0)_\mu^\nu = \delta_{\mu\, N}\delta^{\nu\, N} - \delta_{\mu\, 1}\delta^{\nu\, 1} \,. \end{aligned} \tag{22}$$

In equations (22) the index i goes from 1 to $N-1$, while μ, ν range from 1 to N. Let $\{f^{(k)}\}_{k\in\mathbb{N}}$ be an arbitrary orthonormal basis in $L^2(\mathbb{R}, \mathrm{d}x)$. The vectors

$$f^{(k,\alpha)} \in \mathcal{H} \qquad f_\mu^{(k,\alpha)}(x) = \delta_\mu^\alpha f^{(k)}(x) \qquad k \in \mathbb{N} \quad \alpha = 1, \ldots, N$$

are then an orthonormal basis in $\mathcal{H}$. If $\xi \in \mathbb{R}$, we denote by f_ξ the function f translated by ξ, i.e. $f_\xi(x) = f(x-\xi)$. Let ξ_α denote N arbitrary real numbers and consider the following operators defined on $\mathcal{F}_R^0(\mathcal{H})$:

$$\begin{aligned} J_\alpha^+ &= \int_{\mathbb{R}} \mathrm{d}x\; a^{*\mu}(x)\,(E_\alpha)_\mu^\nu\, a_\nu(x+\xi_\alpha) = \operatorname*{s-lim}_{m\to\infty} \sum_{k=0}^m a^*(f^{(k,\mu)})(E_\alpha)_\mu^\nu a(f_{\xi_\alpha}^{(k,\nu)}) \\ J_\alpha^- &= \int_{\mathbb{R}} \mathrm{d}x\; \tilde{a}^{*\mu}(x)\,(F_\alpha)_\mu^\nu\, \tilde{a}_\nu(x-\xi_\alpha) = \operatorname*{s-lim}_{m\to\infty} \sum_{k=0}^m \tilde{a}^*(f^{(k,\mu)})(F_\alpha)_\mu^\nu \tilde{a}(f_{-\xi_\alpha}^{(k,\nu)}) \\ J_\alpha^0 &= \int_{\mathbb{R}} \mathrm{d}x\; a^{*\mu}(x)\,(H_\alpha)_\mu^\nu\, a_\nu(x) = \operatorname*{s-lim}_{m\to\infty} \sum_{k=0}^m a^*(f^{(k,\mu)})(H_\alpha)_\mu^\nu a(f^{(k,\nu)}) \,. \end{aligned} \tag{23}$$

In these equations a sum over μ, ν is understood. $\{J_\alpha^\pm, J_\alpha^0\}$ are well defined on $\mathcal{F}_R^0(\mathcal{H})$. The simplest way to verify this statement is to establish their action explicitly. One has

$$[J_\alpha^+\varphi]^{(n)}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n) = \sum_{k=1}^n \eta^{k-1}\sum_{\nu_k=1}^N q^{\sum_{h=1}^k \varepsilon(x_h-x_k)\delta^{\mu_k}_{\mu_h}}(E_\alpha)^{\nu_k}_{\mu_k}$$
$$\times\varphi^{(n)}_{\nu_k\mu_1\ldots\hat{\mu}_k\ldots\mu_n}(x_k+\xi_\alpha, x_1,\ldots,\hat{x}_k,\ldots,x_n) \qquad (24)$$

$$[J_\alpha^-\varphi]^{(n)}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n) = \sum_{k=1}^n \eta^{n-k}\sum_{\nu_k=1}^N q^{\sum_{h=k}^n \varepsilon(x_k-x_h)\delta^{\mu_k}_{\mu_h}}(F_\alpha)^{\nu_k}_{\mu_k}$$
$$\times\varphi^{(n)}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_n\nu_k}(x_1,\ldots,\hat{x}_k,\ldots,x_n,x_k-\xi_\alpha) \qquad (25)$$

$$[J_\alpha^0\varphi]^{(n)}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n) = \sum_{k=1}^n\sum_{\nu_k=1}^N (H_\alpha)^{\nu_k}_{\mu_k}\varphi^{(n)}_{\mu_1\ldots\nu_k\ldots\mu_n}(x_1,\ldots,x_n)\,. \qquad (26)$$

These operators are bounded on each $\mathcal{H}_R^n$ and the J_α^0 are selfadjoint. One may check by direct computation that they satisfy the defining relations for the generators of $\mathcal{U}_q(\hat{A}_{N-1})$ and that the central charge is

$$\gamma = \sum_{\alpha=0}^{N-1} J_\alpha^0 = 0\,. \qquad (27)$$

In the case of lattice anyons, a non-trivial central charge can be obtained by means of a Bogoljubov transformation. The naive generalization of this transformation to the continuous case turns out to be ill-defined, so how to get representations with non-zero central charge in the above framework remains an open question. Note that for $\eta = 1$ the operators (23) are written in terms of 'bosonic' anyons. This is different from what happens on the lattice [2–5], where the analogues of $J_\alpha^\pm$ and J_α^0 can be constructed only using 'fermionic' anyons. In this respect our realization shows that this 'hard-core' constraint is essential only for the lattice construction. Equations (24)–(26) also clarify the role of the real parameters $\{\xi\}$: indeed explicit computations show that different choices for $\{\xi\}$ give rise to unitarily inequivalent representations of $\mathcal{U}_q(\hat{A}_{N-1})$. If we put all the $\xi_\alpha = 0$, J_α^+, J_α^- and J_α^0 become multiplicative operators, acting in the space $\mathcal{H}_R^n$ respectively with matrices

$$\mathcal{J}_\alpha^{+(n)} = \sum_{k=1}^n q^{\varepsilon(x_1-x_k)H_\alpha}\otimes q^{\varepsilon(x_2-x_k)H_\alpha}\otimes\cdots\otimes q^{\varepsilon(x_{k-1}-x_k)H_\alpha}\otimes E_\alpha\otimes\mathbb{1}^{\otimes(n-k)}$$
$$\mathcal{J}_\alpha^{-(n)} = \sum_{k=1}^n \mathbb{1}^{\otimes(k-1)}\otimes F_\alpha\otimes q^{\varepsilon(x_{k+1}-x_k)H_\alpha}\otimes q^{\varepsilon(x_{k+2}-x_k)H_\alpha}\otimes\cdots\otimes q^{\varepsilon(x_n-x_k)H_\alpha} \qquad (28)$$
$$\mathcal{J}_\alpha^{0(n)} = \sum_{k=1}^n \mathbb{1}^{\otimes(k-1)}\otimes H_\alpha\otimes\mathbb{1}^{\otimes(n-k)}\,.$$

In this case it is interesting to analyse if there is an interplay between second quantization of the operators J_α^+, J_α^-, J_α^0, i.e. their action on the n-particle space with respect to their action on the one-particle space, and the co-product (19) of the quantum algebra. One may verify that if $x_1 > x_2 > \cdots > x_n$, then $\mathcal{J}_\alpha^{0,\pm(n)}$ is related to $\mathcal{J}_\alpha^{0,\pm(1)}$ by the n-fold nested coproduct (20), while if $x_1 < x_2 < \cdots < x_n$, this connection is given by $\Delta^{(n)}_{q^{-1}}$, where $\Delta_{q^{-1}}$ is obtained by replacing q with q^{-1} in (19) and is again a co-product for $\mathcal{U}_q(\hat{A}_{N-1})$. The nature of this interplay in the intermediate cases, as well as for non-vanishing $\{\xi\}$, is yet to be clarified.

In conclusion, we have explicitly obtained a second quantized representation of the quantum algebras $\mathcal{U}_q(A_N), \mathcal{U}_q(\hat{A}_N)$ on the Fock space of continuous one-dimensional anyons, both of bosonic and fermionic type. Our construction can be easily extended to s-dimensional anyons. The only slightly non-trivial point consists in the replacement of the sign function. In one dimension there is a natural order, which allows for the definition of the sign function. In higher dimensions, one must fix an external vector u and replace $\varepsilon(x)$ by $\varepsilon(x \cdot u)$. This is the algebraic reason for the presence of a 'tail' in two [7] and higher dimensions. The representations that we have obtained depend on arbitrary displacement parameters $\{\xi\}$, and are in general inequivalent for different choices of these parameters. Whether it is possible to find continuous anyonic representations of $\mathcal{U}_q(\hat{A}_N)$ with non-vanishing central charge remains an open question.

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